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Joint weak hazard rate order under non-symmetric copulas

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Abstract: A weak version of the joint hazard rate order, useful to stochastically compare not independent random variables, has been recently defined and studied in [4]. In the present paper, further results on this order are proved and discussed. In particular, some statements dealing with the relationships between the joint weak hazard rate order and other stochastic orders are generalized to the case of non symmetric copulas, and its relations with some multivariate aging notions (studied in [2]) are presented. For this purpose, the new notions of *Generalized Supermigrative* and *Generalized Submigrative* copulas are defined. Other new results, examples and discussions are provided as well.

Keywords: stochastic orders, copulas, dependence notions, generalized supermigrativity

MSC: 60E15, 62H20, 62N05

1 Introduction and useful notions

In the last decades, stochastic comparisons between univariate random variables have been defined and applied in a variety of contexts (see, e.g., the monographs [16, 20] and [3] for detailed descriptions and properties of the main univariate stochastic orders). It is a remarkable fact that most of the univariate stochastic orders considered in the literature are based on comparisons between the marginal distributions of the involved variables, without taking care of their mutual dependence. In fact, in many applied problems one can avoid to consider dependence among alternatives. However, in some cases one has to take it into account, and for this reason a set of alternative bivariate versions of the most well-known stochastic orders have been provided by different authors, like, e.g., in [4, 11, 22]. These versions, which allows to take into account the effects of dependence between the variables to be compared, gave rise to a new class of stochastic comparisons, commonly called *joint stochastic orders*.

A discussion on one of these stochastic comparisons, i.e., the *joint weak hazard rate order*, recently introduced and applied in [4], and in particular on its relationships with the standard hazard rate order, will be presented along this paper. Marginally, the *stochastic precedence order*, which is another well-known comparison that takes into account the mutual dependence among alternatives (see, e.g., [6, 17] and references therein), will be considered and discussed as well.

The following is the formal definition of all the stochastic comparisons discussed in the present paper. Here, the notation $[X|A]$ means the variable whose distribution is the distribution of X given the event A. Also, given the bivariate random vector (X_1, X_2), $\overline{F}$ denotes its joint survival function (i.e., $\overline{F}(t, s) = \mathbb{P}[X_1 > t, X_2 > s]$), $\overline{F}_i$ denotes the survival function of X_i, while f_i and $r_i = f_i/\overline{F}_i$ denote the density and the hazard rate of X_i, whenever it is absolutely continuous.

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Definition 1.1. *Given the random pair (X_1, X_2), we say that X_1 is greater than X_2 in:*

(a) *the* usual stochastic order *(denoted by $X_1 \geq_{st} X_2$) if $E[\phi(X_1)] \geq E[\phi(X_2)]$ for all non-decreasing functions $\phi : \mathbb{R} \to \mathbb{R}$ such that the expectations exists, or, equivalently, if $\overline{F}_1(t) \geq \overline{F}_2(t)$ for all $t \in \mathbb{R}$;*

(b) *the* stochastic precedence order *(denoted by $X_1 \geq_{sp} X_2$) if $\mathbb{P}[X_1 \geq X_2] \geq 1/2$;*

(c) *the* hazard rate order *(denoted by $X_1 \geq_{hr} X_2$) if $[X_1 - t | X_1 > t] \geq_{st} [X_2 - t | X_2 > t]$ for all $t \in \mathbb{R}$ or, equivalently, if the ratio $\overline{F}_1(t)/\overline{F}_2(t)$ is non-decreasing in $t \in \mathbb{R}$;*

(d) *the* joint weak hazard rate order *(denoted by $X_1 \geq_{hr:wj} X_2$) if $[X_1 - t | X_1 > t, X_2 > t] \geq_{st} [X_2 - t | X_1 > t, X_2 > t]$ for all $t \in \mathbb{R}$ or, equivalently, if $\overline{F}(s, t) \leq \overline{F}(t, s)$ for all $t \geq s$, such that $\overline{F}(s, s) > 0$, with $s \in \mathbb{R}$.*

Observe that, even if the term *stochastic order* is used here, the stochastic precedence order and the joint weak hazard rate order are not proper orderings, since they do not satisfy the transitive property. For what it concerns the stochastic precedence order, a discussion and examples showing that it is not transitive may be found, for example, in [7], while a counterexample showing that the joint weak hazard rate order does not satisfy transitivity is given here.

Example 1.1. *Let (X_1, X_2, X_3) be a random vector taking values $(2, 1, 0)$ and $(1, 2, 2)$ respectively with probability $2/3$ and $1/3$. It is easy to verify that both $X_1 \geq_{hr:wj} X_2$ and $X_2 \geq_{hr:wj} X_3$ hold. However, the stochastic inequality $X_1 \geq_{hr:wj} X_3$ does not hold, being, for example, $\mathbb{P}[X_1 > 1/2, X_3 > 3/2] = 1/3 \geq \mathbb{P}[X_1 > 3/2, X_3 > 1/2] = 0$.* □

Also, it should be remarked that when X_1 and X_2 are stochastically independent, then the usual stochastic order implies the precedence order, while, in the case of dependence, it can happens that $X_1 \geq_{st} X_2$ holds even if $X_1 \geq_{sp} X_2$ fails (see [6]). Concerning the hazard rate order and the joint weak hazard rate order, they are equivalent each other when X_1 and X_2 are stochastically independent. However, even if other cases where these two orders are equivalent each other can be provided (see Remark 2.1), in general in case of dependence the hazard rate order and the joint weak hazard rate order are not equivalent (see, e.g., the introductory example presented in [4]).

For this reason, conditions for the hazard rate order to imply the joint weak hazard rate order, or viceversa, are shown in [4]. To describe these conditions, the definitions of some useful notions are recalled. First of all we address the reader e.g. to [18] or [15] for the formal definition of the well-known concept of bivariate copula. See also [10] for a recent survey. We denote by $\mathcal{C}$ the class of all bivariate copulas.

Definition 1.2. *Let (X_1, X_2) be a random pair with joint distribution function F and marginal distribution functions F_1 and F_2. The copula $C : [0, 1]^2 \to [0, 1]$ such that, for all $(x_1, x_2) \in \mathbb{R}^2$,*

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$$

is said to be the connecting copula *of (X_1, X_2). In this case, it also holds*

$$C(u, v) = F(F_1^{-1}(u), F_2^{-1}(v)),$$

for all $u, v \in [0, 1]$.

In a similar way is defined the survival copula of a bivariate random vector (X_1, X_2).

Definition 1.3. *Let (X_1, X_2) be a random pair with joint survival function $\overline{F}$ and marginal survival functions $\overline{F}_1$ and $\overline{F}_2$. The function $\widehat{C} : [0, 1]^2 \to [0, 1]$ such that, for all $(x_1, x_2) \in \mathbb{R}^2$,*

$$\overline{F}(x_1, x_2) = \widehat{C}(\overline{F}_1(x_1), \overline{F}_2(x_2))$$

is said to be the survival copula *of (X_1, X_2). In this case, it also holds*

$$\widehat{C}(u, v) = \overline{F}(\overline{F}_1^{-1}(u), \overline{F}_2^{-1}(v)),$$

for all $u, v \in [0, 1]$.

We observe that from a mathematical viewpoint, survival copulas and connecting copulas turn out in any case to be copulas, and that they both describe the dependence structure of (X_1, X_2). Also, it is well known that if the marginal distributions are continuous then the connecting copula C and the survival copula $\widehat{C}$ are unique. For this reason, we will assume here, and everywhere throughout the paper, continuity of the marginal distributions for the vector (X_1, X_2). We address the readers to the monograph [18] for further details.

The following property of bivariate copulas was firstly introduced by [2], and further studied and applied in dependence analysis in [8, 9], where it has been called *supermigrativity*.

Definition 1.4. *A bivariate copula* $C : [0, 1]^2 \longrightarrow [0, 1]$ *is called supermigrative if it is symmetric, i.e.* $C(u, v) = C(v, u)$ *for every* $(u, v) \in [0, 1]^2$*, and if it satisfies*

$$C(\gamma u, v) \geq C(u, \gamma v) \tag{1.1}$$

for all $u \geq v$ *and* $\gamma \in (0, 1)$*. Viceversa, we say that C is submigrative if the inequality in (1.1) is satisfied in the opposite direction. In the similar way are defined the supermigrativity and submigrativity properties for survival copulas.*

Thus, given any $u_2 \leq u_1 \leq v_1, \leq v_2$ such that $u_1 v_1 = u_2 v_2$, the supermigrativity of a copula C is satisfied if $C(u_1, v_1) \geq C(u_2, v_2)$, i.e., whenever it assumes higher values in correspondence to points (u, v) which are near to the diagonal $v = u$. This means that it has its probability mass mainly concentrated close to the diagonal, and this property can be though of as a positive dependence notion. In fact, as shown in [9], the supermigrativity property satisfies almost all the necessary conditions to be considered a positive dependence notion. For example, any vector (X_1, X_2) having a supermigrative copula (or survival copula) satisfies the *Positive Quadrant Dependence* property (PQD), i.e., it satisfies

$$\mathbb{P}[X_1 > x_1, X_2 > x_2] \geq \mathbb{P}[X_1 > x_1]\mathbb{P}[X_2 > x_2] \text{ for all } (x_1, x_2) \in \mathbb{R}^2.$$

Viceversa, submigrativity can be seen as a negative dependence notion.

Concerning the general meaning of stochastic dependence for a pair of random variables X_1 and X_2 we remind, in particular, that [12]axiomatically defined the set of conditions that X_1 and X_2 should satisfy in order to be positively dependent, i.e, the property that *large* (respectively, *small*) values of X_1 tend to go together with *large* (respectively, *small*) values of X_2 (and the opposite for negative dependence). In any case, notions of positive dependence are typically defined by appropriate inequalities. Corresponding notions of negative dependence can be defined by requiring that such inequalities are reverted. Thus the analysis of both positive and negative dependence is conceptually simple in the case $n = 2$, which this paper is confined to. The panorama about negative dependence is less clear in the case of n variables, with $n > 2$. Anyway, interesting definitions and related results have been given in the literature; see, in particular, the basic paper [5].

The following statement has been proved in [4].

Theorem 1.1. *Let* (X_1, X_2) *be any couple of lifetimes, and let* $\widehat{C}$ *denote its survival copula.*
(a) *If* X_1 *and* X_2 *satisfy* $X_1 \geq_{\text{hr}} X_2$ *and* $\widehat{C}$ *is supermigrative, then* $X_1 \geq_{\text{hr:wj}} X_2$*;*
(b) *If* $X_1 \geq_{\text{hr:wj}} X_2$ *holds, and if* $\widehat{C}$ *is submigrative, then* $X_1 \geq_{\text{hr}} X_2$.

Roughly speaking, recalling that supermigrative property can be though of as a positive dependence notion, the last statement asserts that standard univariate hazard rate order and positive dependence imply the joint weak hazard rate order, while, viceversa, joint weak hazard rate order with negative dependence imply the standard hazard rate order. An example where Theorem 1.1(a) applies is the following (while examples where Theorem 1.1(b) applies may be found in [4]).

Example 1.2. (Conditional independence) *Let* $\Theta \sim \Gamma(\alpha, \beta)$*, and let* (X_1, X_2) *be a vector of exponentially distributed random lifetimes conditionally independent on* Θ*, such that, for* $x_1, x_2 \geq 0$*,*

$$\mathbb{P}[X_1 > x_1, X_2 > x_2 | \Theta = \theta] = e^{-\theta(10x_1 + x_2)}.$$

Observe that, for all $t, s \geq 0$,

$$\overline{F}_1(t) = \mathbb{P}[X_1 > t] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-10\theta t} \theta^{\alpha-1} e^{-\beta\theta} d\theta = \left(\frac{\beta}{\beta + 10t}\right)^\alpha$$

and

$$\mathbb{P}[X_1 > t + s | X_1 > t] = \frac{\mathbb{P}[X_1 > t + s]}{\mathbb{P}[X_1 > t]} = \left(\frac{\beta + 10t}{\beta + 10t + 10s}\right)^\alpha.$$

Similarly,

$$\overline{F}_2(t) = \left(\frac{\beta}{\beta + t}\right)^\alpha \quad \text{and} \quad \mathbb{P}[X_2 > t + s | X_2 > t] = \left(\frac{\beta + t}{\beta + t + s}\right)^\alpha,$$

thus

$$\mathbb{P}[X_1 > t + s | X_1 > t] \leq \mathbb{P}[X_2 > t + s | X_2 > t] \tag{1.2}$$

for all $t, s \geq 0$, i.e., $X_2 \geq_{\text{hr}} X_1$.

Moreover, it holds

$$\begin{aligned}
\mathbb{P}[X_1 > t + s | X_1 > t, X_2 > t] &= \frac{\mathbb{P}[X_1 > t + s, X_2 > t]}{\mathbb{P}[X_1 > t, X_2 > t]} \\
&= \frac{\frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-10\theta t} e^{-10\theta s} e^{-\theta t} \theta^{\alpha-1} e^{-\beta\theta} d\theta}{\frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-10\theta t} e^{-\theta t} \theta^{\alpha-1} e^{-\beta\theta} d\theta} \\
&= \left(\frac{\beta + 10t + t}{\beta + 10t + 10s + t}\right)^\alpha \\
&= \left(\frac{\beta + 11t}{\beta + 11t + 10s}\right)^\alpha,
\end{aligned}$$

and, similarly,

$$\mathbb{P}[X_2 > t + s | X_1 > t, X_2 > t] = \left(\frac{\beta + 11t}{\beta + 11t + s}\right)^\alpha.$$

Clearly,

$$\mathbb{P}[X_1 > t + s | X_1 > t, X_2 > t] \leq \mathbb{P}[X_2 > t + s | X_1 > t, X_2 > t], \tag{1.3}$$

for every $t, s > 0$, i.e., $X_2 \geq_{\text{hr:wj}} X_1$.

In fact, the vector (X_1, X_2) has a Clayton survival copula, defined as

$$\widehat{C}(u, v) = \left(u^{-\frac{1}{\alpha}} + v^{-\frac{1}{\alpha}} - 1\right)^{-\alpha}, \quad u, v \in [0, 1],$$

which satisfies the supermigrative property, thus the last inequality is also proved by inequality (1.2) and Theorem 1.1 (a). □

In general, one can thus observe that the existence of positive dependence strengthens the transition from standard hazard rate order to the corresponding joint one. As a main purpose of this article, in the next section we will treat the case of non-symmetric copulas, providing a generalization of Theorem 1.1 to non–exchangeable cases. Also, coming back to the case of exchangeable survival copulas, in Section 3 we will point out some basic relations between the joint weak hazard rate order and concepts of bivariate ageing for exchangeable lifetimes, that had been presented in [2].

2 Relationships among stochastic orders

In this section we aim to analyze some aspects of the joint weak hazard rate order for pairs of lifetimes (X_1, X_2) whose survival copula $\widehat{C}$ is generally non-symmetric. On this purpose, the following notation will be useful. Let us denote by $\mathcal{S}$ the class of ordered pairs (X_1, X_2) such that $X_1 \geq_{\mathrm{hr}} X_2$, and with $\mathcal{S}(\widehat{C})$ the subclass of $\mathcal{S}$ of the pairs having survival copula $\widehat{C}$. We furthermore denote by $\mathcal{M}$ the class of ordered pairs (X_1, X_2) such that $X_1 \geq_{\mathrm{hr:wj}} X_2$, and with $\mathcal{M}(\widehat{C})$ the subclass of $\mathcal{M}$ of the pairs having survival copula $\widehat{C}$. By using this notation in the case of a symmetric copula, for instance, implication (a) of Theorem 1.1 reads

$$\widehat{C} \text{ supermigrative } \Rightarrow \mathcal{S}(\widehat{C}) \subseteq \mathcal{M}(\widehat{C}).$$

As already noticed in Section 1, we have that, when X_1, X_2 are independent, namely when $\widehat{C}$ is the product copula, then $\mathcal{S}(\widehat{C}) = \mathcal{M}(\widehat{C})$. We shall in particular see that there exist other copulas $\widehat{C}$ for which the same identity holds (see subsequent Remark 2.1). Further properties of the class $\mathcal{M}(\widehat{C})$ will be demonstrated below.

We start by observing that, trivially, the following equivalence holds. In the following statement, and everywhere along the paper, the notation $=_{st}$ means equality in law.

Theorem 2.1. *Let $X_1 =_{st} X_2$. Then $(X_1, X_2) \in \mathcal{M}(\widehat{C})$ iff $\widehat{C}(u, v) \geq \widehat{C}(v, u)$ for all $0 \leq u \leq v \leq 1$.*

Proof. For fixed $t \in \mathbb{R}$ and $s \in \mathbb{R}^+$, let $u = \overline{F}_1(t + s) = \overline{F}_2(t + s)$ and $v = \overline{F}_1(t) = \overline{F}_2(t)$. Then

$$\begin{aligned} \mathbb{P}[X_1 > t + s, X_2 > t] &= \widehat{C}(u, v) \\ &\geq \widehat{C}(v, u) \\ &= \mathbb{P}[X_1 > t, X_2 > t + s], \end{aligned}$$

i.e., $X_1 \geq_{\mathrm{hr:wj}} X_2$.

Viceversa, fix $u, v \in [0, 1]$ such that $u \leq v$. Recalling the assumption of continuity of the marginal distributions (needed to guarantee unicity of connecting and survival copulas, see Section 1), we can assert there exist $t, s \geq 0$ such that $u = \overline{F}_1(t + s) = \overline{F}_2(t + s)$ and $v = \overline{F}_1(t) = \overline{F}_2(t)$. Thus

$$\begin{aligned} \widehat{C}(u, v) &= \mathbb{P}[X_1 > t + s, X_2 > t] \\ &\geq \mathbb{P}[X_1 > t, X_2 > t + s] \\ &= \widehat{C}(v, u). \end{aligned}$$

□

Examples of survival copulas $\widehat{C}$ satisfying the assumption of Theorem 2.1 will be provided later (see, e.g., Example 2.2 and Example 2.3).

The main reason of interest in Theorem 2.1 is in the property required on the survival copula, i.e., the inequality $\widehat{C}(u, v) \geq \widehat{C}(v, u)$ for all $0 \leq u \leq v \leq 1$. As shown later, this property will play a fundamental role in the analysis of relationships between joint weak hazard rate order and other stochastic orders in case of non-symmetric copulas. A first example in this direction is given by the following statement, dealing with relationship with the stochastic precedence order.

Theorem 2.2. *Let (X_1, X_2) be such that $X_1 =_{st} X_2$. Then $X_1 \geq_{hr:wj} X_2$ implies $X_1 \geq_{sp} X_2$.*

Proof. To simplify the notation we give here the proof assuming that the survival copula $\hat{C}$ connecting the pair (X_1, X_2) admits a density $\widehat{c}$. The proof can be easily generalized to not absolutely continuous survival copulas.

Let $X_1 =_{st} X_2$. Then, by Theorem 2.1,

$$\begin{aligned}
X_1 \geq_{hr:wj} X_2 \quad &\Leftrightarrow \quad \hat{C}(u,v) \leq \hat{C}(v,u) \ \forall\, 1 \geq u \geq v \geq 0 \\
&\Rightarrow \quad \lim_{u \to v^+} \frac{\hat{C}(u,v) - \hat{C}(v,v)}{u-v} \leq \lim_{u \to v^+} \frac{\hat{C}(v,u) - \hat{C}(v,v)}{u-v} \ \forall\, 1 \geq v \geq 0 \\
&\Leftrightarrow \quad \int_0^v \hat{c}(v,z)dz \leq \int_0^v \hat{c}(z,v)dz \ \forall\, 1 \geq v \geq 0 \\
&\Rightarrow \quad \int_0^1 \left[\int_0^v \hat{c}(v,z)dz \right] dv \leq \int_0^1 \left[\int_0^v \hat{c}(z,v)dz \right] dv \\
&\Leftrightarrow \quad \int_{\{(u,v)\in[0,1]^2 : u \leq v\}} d\widehat{C}(u,v) \geq 1/2 \\
&\Leftrightarrow \quad X_1 \geq_{sp} X_2,
\end{aligned}$$

where the last equivalence, which holds true whenever $X_1 =_{st} X_2$, follows from Theorem 5 in [6]. □

It should be remarked here that implication $X_1 \geq_{hr:wj} X_2 \Rightarrow X_1 \geq_{sp} X_2$ holds even in case X_1 and X_2 have different distributions, as one can prove by using a proof similar to the one above, just replacing the survival copula $\widehat{C}$ with the joint survival function $\overline{F}$.

The following example shows that the opposite implication can fail. In particular, this conclusion can be achieved by showing that there exist survival copulas satisfying $\int_A d\widehat{C}(u,v) \geq 1/2$ but not necessarily $\widehat{C}(u,v) \geq \widehat{C}(v,u)$ for all $0 \leq u \leq v \leq 1$.

Example 2.1. *Let (X_1, X_2) has survival copula $\widehat{C}$ admitting the density $\widehat{c}$ shown in Figure 1, where $e \in [0, 1/12)$, and where $\widehat{c}(u,v) = 0$ in the white region, $\widehat{c}(u,v) = 4/3$ in the grey region, and $\widehat{c}(u,v) = 4$ in the black region.*

It is easy to observe that, for $e = 0$, it holds $\int_A d\widehat{C}(u,v) = 7/12$ and $\widehat{C}(u,v) \geq \widehat{C}(v,u)$ for all $1 \geq u \geq v \geq 0$, thus both $X_1 \geq_{hr:wj} X_2$ and $X_1 \geq_{sp} X_2$ are satisfied when $X_1 =_{st} X_2$.

Let now $0 < e < 1/12$. It is easy to observe that in this case

$$\int_A d\widehat{C}(u,v) \geq \frac{7}{12} - e > \frac{7}{12} - \frac{1}{12} = \frac{1}{2},$$

thus $X_1 \geq_{sp} X_2$ holds for $X_1 =_{st} X_2$. However, fixing $u' = 3/4 - e$ and $v' = u' + d$, with $d \in (0, 1/4)$, it holds

$$\begin{aligned}
\widehat{C}(v',u') &= \widehat{C}(u',u') + d, \\
\widehat{C}(u',v') &= \widehat{C}(u',u') + \frac{4}{3}\left(\frac{3}{4} - e\right) d = \widehat{C}(u',u') + d - \frac{3}{4} e d.
\end{aligned}$$

Thus, the inequality $\widehat{C}(u,v) \geq \widehat{C}(v,u)$ is no more satisfied for all $1 \geq u \geq v \geq 0$, and inequality $X_1 \geq_{hr:wj} X_2$ does not hold.

□

We now drop the assumption $X_1 =_{st} X_2$, and look for conditions for $X_1 \geq_{hr:wj} X_2$, without necessarily requiring that $\widehat{C}$ is symmetric. For this purpose we provide the following definitions, which extend the notions of supermigrativity and submigrativity to non-symmetric copulas and survival copulas .

Definition 2.1. *A bivariate copula $C : [0,1]^2 \longrightarrow [0,1]$ is said to be* generalized supermigrative *if it satisfies*

a) $C(u,v) \leq C(v,u)$ *for all* $u \geq v$;

b) $C(\gamma u, v) \geq C(u, \gamma v)$ *for all* $u \geq \gamma u \geq v \geq \gamma v$.

Viceversa, we say that C is generalized submigrative *if inequality (b) is satisfied in the opposite direction.*

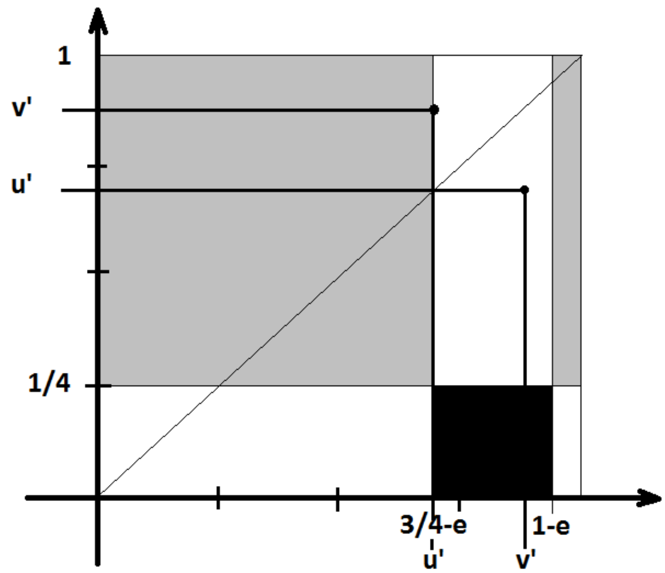

Figure 1: Density of the copula $\widehat{C}$ considered in Example 2.1.

Obviously, symmetric supermigrative (submigrative) copulas are also generalized supermigrative (submigrative) copulas. Examples of non-symmetric generalized supermigrative or generalized submigrative copulas (or survival copulas) will be given later (see Example 2.2 and Example 2.3).

Theorem 1.1 can be now generalized to non-symmetric copulas. More precisely, we have the following statement (which includes Theorem 1.1 as special case).

Theorem 2.3. *Let (X_1, X_2) be any couple of lifetimes, and let $\widehat{C}$ denote its survival copula.*
(a) *If $\widehat{C}$ is generalized supermigrative then $\mathcal{S}(\widehat{C}) \subseteq \mathcal{M}(\widehat{C})$;*
(b) *If $\widehat{C}$ is generalized submigrative then $\mathcal{M}(\widehat{C}) \subseteq \mathcal{S}(\widehat{C})$.*

Proof. We give here only the proof of statement (a), the other being similar. Thus, let us assume that $X_1 \geq_{hr} X_2$. Let $\overline{F}_i$ denotes the survival function of X_i, for $i = 1, 2$. From $X_1 \geq_{hr} X_2$ it follows $X_1 \geq_{st} X_2$, thus also, for $s \leq t$,

$$\overline{F}_2(t) \leq \overline{F}_2(s) \leq \overline{F}_1(t) \leq \overline{F}_1(s) \tag{2.1}$$

or

$$\overline{F}_2(t) \leq \overline{F}_1(t) < \overline{F}_2(s) \leq \overline{F}_1(s). \tag{2.2}$$

Moreover, from $X_1 \geq_{hr} X_2$ also follows, for $s \leq t$,

$$\overline{F}_2(s)\overline{F}_1(t) \geq \overline{F}_2(t)\overline{F}_1(s). \tag{2.3}$$

Assume that (2.1) holds. Then, for $s \leq t$,

$$\begin{aligned} \overline{F}(t, s) &= \widehat{C}(\overline{F}_1(t), \overline{F}_2(s)) \\ &= \widehat{C}\Big(\overline{F}_1(t), \overline{F}_2(t)\frac{\overline{F}_2(s)}{\overline{F}_2(t)}\Big). \end{aligned}$$

Let us denote

$$v = \overline{F}_2(t)\frac{\overline{F}_2(s)}{\overline{F}_2(t)}, \quad u = \overline{F}_1(t)\frac{\overline{F}_2(s)}{\overline{F}_2(t)}, \quad \gamma = \frac{\overline{F}_2(t)}{\overline{F}_2(s)}.$$

Then

$$\begin{aligned}\overline{F}(t,s) &= \widehat{C}(\gamma u, v)\\ &\geq \widehat{C}(u, \gamma v)\\ &= \widehat{C}\big(\overline{F}_1(t)\frac{\overline{F}_2(s)}{\overline{F}_2(t)}, \overline{F}_2(t)\big)\\ &\geq \widehat{C}\big(\overline{F}_1(s), \overline{F}_2(t)\big)\\ &= \overline{F}(s,t),\end{aligned}$$

where the first inequality follows from generalized supermigrativity of $\widehat{C}$ and by (2.1), while the second one from (2.3).

Assume now that (2.2) holds. Then, for $s \leq t$,

$$\begin{aligned}\overline{F}(t,s) &= \widehat{C}(\overline{F}_1(t), \overline{F}_2(s))\\ &\geq \widehat{C}(\overline{F}_2(s), \overline{F}_1(t))\end{aligned}$$

by inequality (a) in Definition 2.1. Denote now

$$v = \overline{F}_1(t), \quad u = \overline{F}_1(s), \quad \gamma = \frac{\overline{F}_2(s)}{\overline{F}_1(s)}.$$

Then

$$\begin{aligned}\overline{F}(t,s) &\geq \widehat{C}\big(\overline{F}_1(s)\frac{\overline{F}_2(s)}{\overline{F}_1(s)}, \overline{F}_1(t)\big)\\ &= \widehat{C}(\gamma u, v)\\ &\geq \widehat{C}(u, \gamma v)\\ &= \widehat{C}\big(\overline{F}_1(s), \frac{\overline{F}_2(s)}{\overline{F}_1(s)}\overline{F}_1(t)\big)\\ &\geq \widehat{C}\big(\overline{F}_1(s), \overline{F}_2(t)\big)\\ &= \overline{F}(s,t),\end{aligned}$$

where the second inequality follows from from generalized supermigrativity of $\widehat{C}$ and inequality (2.2), while the third from (2.3).

In both cases we have $\overline{F}(t,s) \geq \overline{F}(s,t)$ for all $s \leq t$, thus $X_1 \geq_{\text{hr:wj}} X_2$. □

We now provide some examples of non-symmetric survival copulas satisfying the generalized supermigrative property. A first example is given by non-symmetric Marshall-Olkin copulas, described below.

Example 2.2. *Let the survival copula $\widehat{C}$ be a Marshall-Olkin copula, i.e., be defined as*

$$\widehat{C}(u,v) = \begin{cases} u^{1-\alpha}v & u^\alpha \geq v^\beta \\ uv^{1-\beta} & u^\alpha < v^\beta \end{cases}$$

with $1 > \beta > \alpha > 0$. Observe that $\widehat{C}$ is not absolutely continuous, having a singularity on the curve $v = u^{\alpha/\beta}$.

Let us first prove that $\widehat{C}(u,v) \leq \widehat{C}(v,u)$ holds for all $u \geq v$ (i.e., for all points (u,v) below the diagonal of the unit square, thus also below the singularity). For it, we should consider two cases.
i) Assume that (v,u) is such that $u^\beta > v^\alpha$, i.e., that (v,u) is above the singularity. Then $\widehat{C}(v,u) = vu^{1-\beta}$ and $\widehat{C}(u,v) = u^{1-\alpha}v$, and the inequality is satisfied by $\beta \geq \alpha$.
ii) Assume that (v,u) is such that $u^\beta \leq u^\alpha$, i.e., that (v,u) is below the singularity. Then $\widehat{C}(v,u) = v^{1-\alpha}u$ and $\widehat{C}(u,v) = u^{1-\alpha}v$, and the inequality is satisfied by $u \geq v$.

Let us now prove that $\widehat{C}(\gamma u, v) \geq \widehat{C}(u, \gamma v)$ holds for all $1 \geq u \geq \gamma u \geq v \geq \gamma v \geq 0$. For it, observe that both $(\gamma u, v)$ and $(u, \gamma v)$ belong to the region below the diagonal, i.e., below the singularity, thus $\widehat{C}(\gamma u, v) = \gamma^{1-\alpha}u^{1-\alpha}v$ and the inequality follows from $\gamma \leq 1$. □

Remark 2.1. *Consider a vector (X_1, X_2) defined by*

$$X_1 = \min(Y, Y_1), \quad X_2 = \min(Y, Y_2),$$

where $Y \sim Exp(\lambda)$, $Y_1 \sim Exp(\lambda_1)$ and $Y_2 \sim Exp(\lambda_2)$ are three independent and exponentially distributed random variables. Whenever $\lambda_1 \leq \lambda_2$, we have $X_1 \geq_{hr} X_2$, because the hazard rates of X_1 and X_2 are $\lambda_1 + \lambda$ and $\lambda_2 + \lambda$, respectively. Since vector (X_1, X_2) has the Marshall-Olkin copula described in Example 2.2 (see, [18], pp. 52-53), relation $X_1 \geq_{hr:wj} X_2$ follows from Theorem 2.3(a).

Actually, this assertion can be also directly proved and strengthen just observing that, for this vector (X_1, X_2), it holds

$$\mathbb{P}[X_i > t + s | X_1 > t, X_2 > t] = \mathbb{P}[X_i > t + s | X_i > t] = \mathbb{P}[X_i > s] \tag{2.4}$$

for all $s, t \geq 0$ (see [14]). Thus the inequalities $X_1 \geq_{st} X_2$, $X_1 \geq_{hr} X_2$ and $X_1 \geq_{hr:wj} X_2$ become equivalent, even if X_1 and X_2 are not independent. In other words, this example shows that the standard hazard rate order and the joint weak hazard order can both hold true even for survival copulas $\widehat{C}$ different than the copula of independence.

□

Remark 2.2. *Example 2.2 can be restated assuming the variables Y, Y_1 and Y_2 to be any independent random lifetimes, not necessarily exponentially distributed, thus letting the survival copula $\widehat{C}$ be the Generalized Marshall-Olkin copula described in [13], Equation (2.3). In this case one has*

$$\mathbb{P}[X_i > t + s | X_1 > t, X_2 > t] = \mathbb{P}[X_i > t + s | X_i > t], \tag{2.5}$$

for all $s, t \geq 0$ (see [13] for details). Even in this case the inequalities

$$\mathbb{P}[X_1 > t + s | X_1 > t] \geq \mathbb{P}[X_2 > t + s | X_2 > t]$$

and

$$\mathbb{P}[X_1 > t + s | X_1 > t, X_2 > t] \geq \mathbb{P}[X_2 > t + s | X_1 > t, X_2 > t],$$

i.e., $X_1 \geq_{hr} X_2$ and $X_1 \geq_{hr:wj} X_2$, become equivalent. Thus, in this case it holds $\mathcal{S}(\widehat{C}) = \mathcal{M}(\widehat{C})$ even if X_1 and X_2 are not independent. However, now the equivalence between the standard hazard rate, the joint weak hazard rate and the usual stochastic ($\geq_{st}$) orders no longer applies, since the two conditional probabilities in (2.5) are no more equal to $\mathbb{P}[X_i > s]$, as it is in (2.4). □

The following is a further example of copula satisfying generalized supermigrativity.

Example 2.3. *Let the survival copula $\widehat{C}$ be a member of the the copulas defined in [19], i.e., let $\widehat{C}$ be a generalization of the FGM copula defined as*

$$\widehat{C}(u, v) = uv + \rho u^{\beta_1} v^{\beta_2} (1 - u)^{\alpha_1} (1 - v)^{\alpha_2} \tag{2.6}$$

with $\beta_i, \alpha_i \geq 1$, $i = 1, 2$, and $0 \leq \rho \leq 1$. In case $\beta_1 \leq \beta_2$ and $\alpha_1 \geq \alpha_2$ then generalized supermigrativity holds.

In fact, for inequality (a) in Definition 2.1 it is easy to observe that $\widehat{C}(u, v) \leq \widehat{C}(v, u)$ is satisfied iff $u^{\beta_1-\beta_2}(1 - u)^{\alpha_1-\alpha_2} \leq v^{\beta_1-\beta_2}(1 - v)^{\alpha_1-\alpha_2}$, i.e., iff $u \geq v$, while inequality (b) follows from the fact that

$$\begin{aligned}\widehat{C}(\gamma u, v) \geq \widehat{C}(u, \gamma v) \quad &\Leftrightarrow \quad \rho u^{\beta_1} v^{\beta_2} \left[\gamma^{\beta_1}(1 - \gamma u)^{\alpha_1}(1 - v)^{\alpha_2} - \gamma^{\beta_2}(1 - u)^{\alpha_1}(1 - \gamma v)^{\alpha_2}\right] \geq 0 \\ &\Leftrightarrow \quad \gamma^{\beta_1-\beta_2} \left[\frac{(1 - \gamma u)}{(1 - u)}\right]^{\alpha_1} \geq \left[\frac{(1 - \gamma v)}{(1 - v)}\right]^{\alpha_2},\end{aligned}$$

and the latter follows from

$$\gamma^{\beta_1-\beta_2} \geq 1 \quad \text{and} \quad \frac{(1 - \gamma u)}{(1 - u)} \geq \frac{(1 - \gamma v)}{(1 - v)},$$

which are satisfied when $\gamma \leq 1$ and $u \geq v$. □

An example of application of Theorem 2.3 is now given.

Example 2.4. *Consider two components having dependent lifetimes X_1 and X_2, respectively, and assume identical exponential distribution with rate λ for both. Assume that a task should be performed by both components up to a fixed time T, and then continued by just one of them. Having the same hazard rate, i.e., being*

$$\mathbb{P}[X_1 - T > s | X_1 > T] = \mathbb{P}[X_2 - T > s | X_2 > T] \;\; \forall T, s \geq 0, \tag{2.7}$$

a designer can assume that, when the two components age together, there is no difference in choosing the component that will conclude the task after the fixed time T, if both components have survived up to T, whatever T is. Unfortunately, this assertion can be wrong, since equality (2.7) does not imply

$$\mathbb{P}[X_1 - T > s | X_1 > T, X_2 > T] = \mathbb{P}[X_2 - T > s | X_2 > T, X_2 > T] \;\; \forall T, s \geq 0, \tag{2.8}$$

that is, the residual lifetimes of the components at T, given that both components work at T, can be different. Moreover, in case the inequality in (2.8) is not satisfied, nothing can be asserted in general on the direction of the corresponding inequality. Assume for example, that the survival copula $\widehat{C}$ of (X_1, X_2) is the copula defined in (2.6), with $\rho = 1, \beta_1 = \alpha_1 = 2$ and $\beta_2 = \alpha_2 = 1$ (so that condition $\beta_1 \leq \beta_2$ and $\alpha_1 \geq \alpha_2$ fail). Then, as it can be verified with a direct computation (see Figure 2), the inequality in both directions can holds, depending on $0 \leq s \leq T$. However, in case $\beta_1 \leq \beta_2$ and $\alpha_1 \geq \alpha_2$ then $\widehat{C}$ is generalized supermigrate, thus by Theorem 2.3(a) it

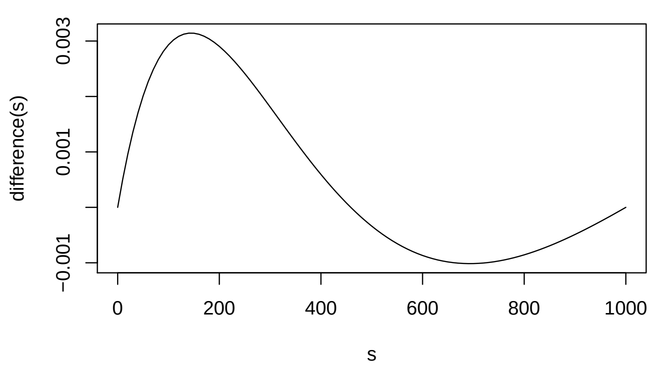

Figure 2: Plot of the difference $\mathbb{P}[X_1 - T > s | X_1 > T, X_2 > T] - \mathbb{P}[X_2 - T > s | X_2 > T, X_2 > T]$ for $T = 1000$ when the vector (X_1, X_2) has exponentially distributed margins, with $E[X_i] = 1000$, and survival copula defined as in (2.6), with $\rho = 1, \beta_1 = \alpha_1 = 2$ and $\beta_2 = \alpha_2 = 1$.

holds $[X_1 - T | X_1 > T, X_2 > T] \geq_{\text{st}} [X_2 - T | X_1 > T, X_2 > T]$, and the designer can choose the first component to conclude the task after time T. □

Similar examples can be provided in other applicative contexts, like in insurance, letting X_1 and X_2 be two dependent risks, and having to compare truncated random claims (where T is the truncation level), or in medicine, letting X_1 and X_2 be two competing risks.

Concerning Theorem 1.1, it is then rather interesting to understand why positive dependence plays in favor of $\mathcal{S}(\widehat{C}) \subseteq \mathcal{M}(\widehat{C})$, i.e., in favor of the implication between inequality $X_1 \geq_{\text{hr}} X_2$ and inequality $X_1 \geq_{\text{hr:wj}} X_2$ (or the viceversa in case of negative dependence). The heuristic interpretation of this fact is given here.

Recalling the definitions of hazard rate order and joint weak hazard rate order, and the intuitive meaning of positive dependence, one can try to write a chain of stochastic inequalities showing the assertion, as follows. Let $t \in \mathbb{R}$, then

$$\begin{aligned}[X_1 - t|\, X_1 > t, X_2 > t] &\geq_{st} [X_1 - t|\, X_1 > t] \\ &\geq_{st} [X_2 - t|\, X_2 > t] \\ &\leq_{st} [X_2 - t|\, X_1 > t, X_2 > t],\end{aligned}$$

where the first inequality follows by the positive dependence existing between X_1 and X_2 (informally, because large values of X_1 tend to go together with large values of X_2), the second one by assumption $X_1 \geq_{hr} X_2$, and the third one again by positive dependence. This chain of inequalities can not be used to prove the assertion that $X_1 \leq_{hr:wj} X_2$, since the last one is in the wrong direction. However, by comparing the inequalities

$$[X_1 - t|\, X_1 > t] \leq_{st} [X_1 - t|\, X_1 > t, X_2 > t]$$

and

$$[X_2 - t|\, X_2 > t] \leq_{st} [X_2 - t|\, X_1 > t, X_2 > t],$$

one can argue that under some suitable conditions of positive dependence the first one is in some sense *stronger* than the second one, being the additional conditional event $\{X_2 > t\}$ *stronger*, in some stochastic sense, than $\{X_1 > t\}$, since $X_1 \geq_{hr} X_2$. This is actually the case if the structure of dependence among X_1 and X_2, i.e., the survival copula $\widehat{C}$, satisfy the generalized supermigrativity property: under this assumption, the last inequality is counterbalanced by the first one.

Because of Theorem 2.3, and its heuristic interpretation, one can conjecture that there exist conditions of positive dependence strong enough to let the inequality $X_1 \geq_{\mathrm{hr:wj}} X_2$ be satisfied under conditions weaker than $X_1 \geq_{\mathrm{hr}} X_2$. In the following two examples, we in particular present special models of dependence for which the relation $X_1 \geq_{\mathrm{hr:wj}} X_2$ is equivalent to $X_1 \geq_{\mathrm{st}} X_2$, and where $X_1 \geq_{\mathrm{hr}} X_2$ can fails. In the first example we analyze the case of maximal positive dependence (i.e. comonotonicity).

Example 2.5. *Let $X_2 = \phi(X_1)$, being ϕ any bijective and differentiable increasing function. Then $X_1 \geq_{\mathrm{hr:wj}} X_2$ iff*

$$\overline{F}(t, s) = \mathbb{P}[X_1 > \max(t, h(s))] \geq \mathbb{P}[X_1 > \max(h(t), s)] = \overline{F}(s, t), \quad \forall s \leq t,$$

where $h = \phi^{-1}$ is the inverse of ϕ.

Let $\phi(t) \leq t$ for every $t \geq 0$ (thus also $X_1 \geq_{\mathrm{st}} X_2$). Under this conditions we have $t \leq h(t)$ and $s \leq h(s) \leq h(t)$ for all $0 \leq s \leq t$, thus also $\max(t, h(s)) \leq h(t) = \max(h(t), s)$. It follows

$$\overline{F}(t, s) = \mathbb{P}[X_1 > \max(t, h(s))] \geq \mathbb{P}[X_1 > h(t)] = \mathbb{P}[X_1 > \max(h(t), s)] = \overline{F}(s, t), \quad \forall s \leq t,$$

i.e., $X_1 \geq_{\mathrm{hr:wj}} X_2$.

On the other hand, still assuming $X_2 = \phi(X_1)$ for an increasing ϕ, stronger conditions on ϕ are required for the inequality $X_1 \geq_{\mathrm{hr}} X_2$. Denoted with r_i the hazard rate of X_i, it holds $r_2(t) = r_1(h(t))h'(t)$, where h is the inverse ϕ^{-1}. Thus, $X_1 \geq_{\mathrm{hr}} X_2$ iff $r_1(t) \leq r_1(h(t))h'(t)$ for all $t \geq 0$. Now, if X_1 is exponentially distributed with rate λ, such inequality is satisfied only for $h'(t) \geq 1$, i.e., for $\phi'(t) \leq 1$ for all t. This is actually possible, but it is a stronger condition than $\phi(t) \leq t$, which is the one required for $X_1 \geq_{\mathrm{st}} X_2$ and $X_1 \geq_{\mathrm{hr:wj}} X_2$. Thus, under maximal positive dependence both $X_1 \geq_{\mathrm{hr:wj}} X_2$ and $X_1 \geq_{\mathrm{hr}} X_2$ can hold true, but, in some cases, it can be satisfied only the inequality $X_1 \geq_{\mathrm{hr:wj}} X_2$. □

We present next a further example where the relation $X_1 \geq_{\mathrm{st}} X_2$ is equivalent to $X_1 \geq_{\mathrm{hr:wj}} X_2$. It is interesting to remark in this respect that we do not hinge here on conditions of positive dependence.

Example 2.6. Load-sharing models with time-homogeneous failure parameters

In the case of absolute continuity, the joint probability law of n non-negative random variables $X_1, \ldots, X_n$ can be described by means of the family of its Multivariate Conditional Hazard Rates *(MCHR) functions. Such a description is alternative but mathematically equivalent to the one expressed in terms of $f_{X_1,\ldots,X_n}$, the joint density function of $(X_1, \ldots, X_n)$. See, e.g., the recent review paper [21] and references cited therein.*

In a sense, the MCHR functions arise as direct extensions of the univariate concept of hazard rate function for a single non-negative random variable X. For our purposes, we can limit ourselves to formulating the definition of MCHR functions for the case of $n = 2$ *non-negative variables* X_1, X_2. *In this case the family of the MCHR functions is* $\mathcal{L} = \{\lambda_1^{(0)}(t), \lambda_2^{(0)}(t), \lambda_1^{(1)}(t,x), \lambda_2^{(1)}(t,x)\}$, *where*

$$\lambda_1^{(0)}(t) := \lim_{\Delta t \to 0^+} \frac{\mathbb{P}[X_1 > t + \Delta t | X_1 > t, X_2 > t]}{\Delta t},$$

$$\lambda_2^{(0)}(t) := \lim_{\Delta t \to 0^+} \frac{\mathbb{P}[X_2 > t + \Delta t | X_1 > t, X_2 > t]}{\Delta t},$$

and, for $0 < x < t$,

$$\lambda_1^{(1)}(t,x) := \lim_{\Delta t \to 0^+} \frac{\mathbb{P}[X_1 > t + \Delta t | X_1 > t, X_2 = x]}{\Delta t}$$

$$\lambda_2^{(1)}(t,x) := \lim_{\Delta t \to 0^+} \frac{\mathbb{P}[X_2 > t + \Delta t | X_2 > t, X_1 = x]}{\Delta t}.$$

The assumption of absolute continuity of the joint distribution is essential since it simultaneously guarantees the meaningfulness of the conditional probabilities appearing above and the needed condition of no-tie*:*

$$\mathbb{P}[X_1 \neq X_2] = 1.$$

The functions $\lambda_1^{(0)}(t), \lambda_2^{(0)}(t), \lambda_1^{(1)}(t,x)$ *and* $\lambda_2^{(1)}(t,x)$ *belonging to* $\mathcal{L}$ *can be obtained in terms of the joint density function* f_{X_1,X_2} *of* (X_1, X_2). *One can also check that, viceversa,* f_{X_1,X_2} *can be recovered from the knowledge of the family* $\mathcal{L}$. *The description of a joint distribution in terms of* $\mathcal{L}$ *turns out, furthermore, to be very efficient in the analysis of some problems of applied probability and in defining appropriate models of dependence. In particular, special models arise from the condition that* $\lambda_1^{(1)}(t,x)$ *and* $\lambda_2^{(1)}(t,x)$ *do not depend on x. The corresponding models can be called models of Load-Sharing. See, e.g., [24] or the monograph [23] and references cited therein for more details and further remarks. In particular we can consider the time-homogeneous case defined by the constants*

$$\lambda_1^{(0)}(t) = \lambda_1^{(0)}, \ \lambda_2^{(0)}(t) = \lambda_2^{(0)}, \ \lambda_1^{(1)}(t,x) = \lambda_1^{(1)}, \ \lambda_2^{(1)}(t,x) = \lambda_2^{(1)}$$

In this case we have, for all $t, s \geq 0$,

$$\mathbb{P}[X_1 > t + s | X_1 > t, X_2 > t] = \mathbb{P}[X_1 > s],$$

and

$$\mathbb{P}[X_2 > t + s | X_1 > t, X_2 > t] = \mathbb{P}[X_2 > s].$$

Thus, the stochastic order $X_1 \geq_{\mathrm{st}} X_2$, *which is satisfied when* $\lambda_2^{(0)} \geq \lambda_1^{(0)}$ *and* $\lambda_2^{(1)} \geq \lambda_1^{(1)}$, *is maintained under the conditioning* $\{X_1 > t, X_2 > t\}$, *so that* $X_1 \geq_{\mathrm{st}} X_2 \Leftrightarrow X_1 \geq_{\mathrm{hr:wj}} X_2$. *Moreover, observe that conditions*

$$\lambda_1^{(0)} \leq \lambda_1^{(1)}, \lambda_2^{(0)} \leq \lambda_2^{(1)}$$

give raise to a positive dependence, while the conditions

$$\lambda_1^{(0)} \geq \lambda_1^{(1)}, \lambda_2^{(0)} \geq \lambda_2^{(1)}$$

give raise to a negative dependence.

Note that if the inequalities $\lambda_1^{(1)} \leq \lambda_2^{(1)} \leq \lambda_1^{(0)} \leq \lambda_2^{(0)}$ *hold, recalling that* $X_1 \geq_{\mathrm{hr}} X_2 \Rightarrow X_1 \geq_{\mathrm{st}} X_2$, *then one has a case where* $X_1 \geq_{\mathrm{hr}} X_2 \Rightarrow X_1 \geq_{\mathrm{hr:wj}} X_2$ *holds true for negatively dependent random variables.* □

3 Relationships with multivariate aging notions

The notion of supermigrative copula had originated from the analysis of some questions concerning the concept of *aging* for a pair of exchangeable lifetimes. In the previous section we saw that the extension of supermigrativity to non-exchangeable copulas can be relevant in the analysis of the $\geq_{\mathrm{hr:wj}}$ property. In this vein, we point out in this section some relations between $\geq_{\mathrm{hr:wj}}$ and the Bivariate Increasing Hazard Rate property that will be recalled below.

First, we recall the definition of IHR property, which is a well-known notion used in the description of the reliability of engineering systems (see, e.g., [1] for details and examples of application).

Definition 3.1. *A non-negative random variable X is said to have* Increasing Hazard Rate*, shortly* IHR*, if it satisfies*

$$[X - t|X > t] \geq_{\mathrm{st}} [X - s|X > s] \;\; \forall\, 0 \leq t \leq s, \tag{3.1}$$

or, equivalently, if $\overline{F}(t+s)/\overline{F}(t)$ is non-increasing in $t \geq 0$ for all $s \geq 0$, where $\overline{F}$ denotes its survival function.

It is useful to observe that, as shown in Theorem 1.B.38(iii) in [20], the following relation holds between IHR notion and hazard rate order: a random lifetime X has IHR if, and only if, $X + s \geq_{\mathrm{hr}} X + t$ whenever $t \leq s$.

A corresponding bivariate notion, recalled here, has been defined in [2].

Definition 3.2. *A couple of exchangeable non-negative random variables (X_1, X_2) is said to have* Bivariate Increasing Hazard Rate*, shortly* B–IHR*, if it satisfies*

$$[X_1 - t|X_1 > t, X_2 > s] \geq_{st} [X_2 - s|\, X_1 > t, X_2 > s] \;\; \forall\, 0 \leq t \leq s, \tag{3.2}$$

or, equivalently, if their joint survival function $\overline{F}$ is Schur-concave.

The following statement provides a relation between B-IHR and joint weak hazard rate order.

Theorem 3.1. *Given the couple (X_1, X_2) of exchangeable lifetimes, if $X_1 + s \geq_{hr:wj} X_2$ for every $s \geq 0$, then it satisfies the B-IHR property.*

Proof. Fix any t_1, t_2 such that $0 \leq t_1 \leq t_2$. By letting $\delta = t_2 - t_1$ it holds:

$$\begin{aligned}
[X_1 - t_1|\, X_1 > t_1, X_2 > t_2] \quad &=_{st} \quad [X_1 - (t_2 - \delta)|\, X_1 + \delta > t_2, X_2 > t_2] \\
&=_{st} \quad [X_1 + \delta - t_2|\, X_1 + \delta > t_2, X_2 > t_2] \\
&\geq_{st} \quad [X_2 - t_2|\, X_1 + \delta > t_2, X_2 > t_2] \\
&=_{st} \quad [X_2 - t_2|\, X_1 > t_1, X_2 > t_2], ,
\end{aligned}$$

where the inequality follows by assumption $X_1 + s \geq_{hr:wj} X_2$ for all $s \geq 0$. □

Relations between univariate aging, bivariate aging and dependence properties have been extensively studied in [2]. Among other results, the following relationship between IHR, B-IHR notions and supermigrativity property follows from Theorem 5.2(1) in [2]. Concerning the language and notation used in that paper, we observe that supermigrativity is denoted as $\mathcal{P}_+^3$ in there and that $\mathcal{P}$-positive 2-aging reduces to Bivariate IHR when one refers to the dependence property of supermigrativity.

Theorem 3.2. *Let (X_1, X_2) be a couple of exchangeable lifetimes, and let $\widehat{C}$ denote its survival copula. If the margins X_1 and X_2 satisfy the IHR univariate aging notion and $\widehat{C}$ is supermigrative, then (X_1, X_2) satisfies the bivariate ageing notion B-IHR.*

As a consequence of the Theorem 3.1, we immediately obtain that Theorem 3.2 above follows as a corollary of Theorem 2.3. In fact, assume that (X_1, X_2) has identically distributed and IHR margins, and that the corresponding survival copula $\widehat{C}$ is supermigrative. Then, from the IHR property of X_1 and Theorem 1.B.38(iii) in [20] one has

$$X_1 + s \geq_{hr} X_1 \quad \forall\, s \geq 0,$$

which in turn implies

$$X_1 + s \geq_{hr} X_2 \quad \forall\, s \geq 0,$$

by exchangeability of X_1 and X_2. Now, by supermigrativity (thus also generalized supermigrativity) of $\widehat{C}$, and applying Theorem 2.3, it follows

$$X_1 + s \geq_{hr:wj} X_2 \quad \forall\, s \geq 0,$$

i.e., that (X_1, X_2) satisfy the B-IHR property by Theorem 3.1.

A statement describing relations between the B-IHR property and the joint weak hazard rate order, in the opposite direction with respect to what stated in Theorem 2.3, is now given. Observe that the assumption on the function ϕ considered here is satisfied, for example, by any differentiable increasing concave function ϕ such that $\phi'(0) \leq 1$.

Theorem 3.3. *Given the couple (X_1, X_2) of exchangeable lifetimes, if it satisfies the B-IHR property then $X_1 \geq_{hr:wj} \phi(X_2)$ for any non negative increasing function ϕ which is subadditive and such that $\phi(t) \leq t$ for all $t \geq 0$.*

Proof. Let $\overline{H}$ denotes the survival function of $\phi(X_2)$, i.e., let $\overline{H}(t) = \overline{G}(\phi^{-1}(t))$, where $\overline{G}$ is the survival function of the margins of X_i. Since $\phi(t) \leq t$ and $X_1 =_{st} X_2$, it clearly holds $X_1 \geq_{st} \phi(X_2)$. From subadditivity of ϕ follows the superadditivity of $\phi^{-1}(t) = \overline{G}^{-1}(\overline{H}(t))$, which, in turns, implies, for all $t, w \geq 0$,

$$\begin{aligned} \overline{G}^{-1}(\overline{H}(t)) + w &\leq \overline{G}^{-1}(\overline{H}(t)) + \overline{G}^{-1}(\overline{H}(w)) \\ &\leq \overline{G}^{-1}(\overline{H}(t + w)), \end{aligned} \tag{3.3}$$

where the first inequality follows from $w \leq \overline{G}^{-1}(\overline{H}(w))$ for all w (i.e., $X_1 \geq_{st} \phi(X_2)$), while the second one follows from superadditivity of $\phi^{-1}(t)$.

Observe now that the $B - IHR$ property of (X_1, X_2) is equivalent to

$$\mathbb{P}[X_1 > t + w \mid X_1 > t, X_2 > t'] \geq \mathbb{P}[X_2 > t' + w \mid X_1 > t, X_2 > t'],$$

for all $w \geq 0$ and $t' \geq t \geq 0$, which, in turns, is equivalent to

$$\mathbb{P}[X_1 > t + w \mid X_1 > t, \phi(X_2) > \phi(t')] \geq \mathbb{P}[\phi(X_2) > \phi(t' + w) \mid X_1 > t, \phi(X_2) > \phi(t')]. \tag{3.4}$$

Let now $t = \phi(t') = \overline{H}^{-1}(\overline{G}(t')) \leq t'$. By (3.4) and (3.3) (i.e., by $\phi^{-1}(t) + w \leq \phi^{-1}(t + w)$) we get

$$\begin{aligned} \mathbb{P}[X_1 > t + w \mid X_1 > t, \phi(X_2) > t] &\geq \mathbb{P}[\phi(X_2) > \phi(\phi^{-1}(t) + w) \mid X_1 > t, \phi(X_2) > t] \\ &\geq \mathbb{P}[\phi(X_2) > t + w \mid X_1 > t, \phi(X_2) > t], \end{aligned}$$

for all $t, w \geq 0$, i.e., $X_1 \geq_{hr:wj} \phi(X_2)$. □

As an immediate example of application of Theorem 3.3, consider an exchangeable vector (X_1, X_2) whose marginal distributions are IHR, and assume its survival copula is of Archimedean type, having log-convex generator. As shown in [2], in this case (X_1, X_2) satisfies the B-IHR bivariate aging property, thus, by previous statement, one has $X_1 \geq_{hr:wj} \phi(X_2)$ for any differentiable increasing concave function ϕ such that $\phi'(0) \leq 1$.

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